ACADEMIC PRESS

Available online at www.sciencedirect.com

SCIENCE DIRECT®

Journal of Algebra 256 (2002) 194–210

JOURNAL OF Algebra

www.academicpress.com

Neat idempotents and tiled orders having large global dimension

Hisaaki Fujita

Institute of Mathematics, University of Tsukuba, Tsukuba Ibaraki, 305-8571 Japan

Received 26 February 2002

Communicated by Kent R. Fuller

Abstract

We study neat primitive idempotents in a semiperfect Noetherian ring, and as an application, we improve an example of a tiled order having large global dimension given by Jansen and Odenthal. Moreover, another two tiled orders having large global dimension are added and two questions on tiled orders of finite global dimension are posed.

Let R be a ring with identity and e an idempotent in R. It is a fundamental problem in ring theory to study relationships between R and eRe for a suitable e.

Let D be a discrete valuation ring with quotient field K. In [JO], Jansen and Odenthal found a tiled D-order having large global dimension, i.e., for every even integer $N \geqslant 8$, they constructed a tiled D-order in the full $N \times N$ matrix ring $\mathrm{M}_N(K)$ whose global dimension is $2N - 8$. In this paper, we study some properties of *neat* primitive idempotents in a semiperfect Noetherian ring (see Section 1) and we improve their example, i.e., starting from $N = 6$, we construct tiled D-orders Γ_N in $\mathrm{M}_N(K)$ inductively, and we show that gl.dim $\Gamma_6 =$ gl.dim $\Gamma_7 = 5$ and gl.dim $\Gamma_N = 2N - 8$ for *all* $N \geqslant 8$, using properties of neat idempotents.

We now recall some facts on tiled orders having finite global dimension. In his study of global dimension of orders [T1,T2], Tarsy conjectured that if Λ is

E-mail address: fujita@math.tsukuba.ac.jp.

0021-8693/02/$ – see front matter
PII: S0021-8693(02)00163-1

a D-order of finite global dimension in $\mathrm{M}_n(K)$, global dimension of Λ is bounded by $n-1$. In [J2], Jategaonkar showed that there are only finitely many tiled D-orders of finite global dimension in $\mathrm{M}_n(K)$ for a fixed n, so that there is a upper bound of finite global dimension. As a strategy to prove Tarsy's conjecture for tiled D-orders, Jategaonkar conjectured that if Λ is a tiled D-order of finite global dimension then there exists a primitive idempotent $e_n \in \Lambda$ such that $(1-e_n)\Lambda e_n$ or $e_n\Lambda(1-e_n)$ is $(1-e_n)\Lambda(1-e_n)$-projective. In some special cases, both conjectures were settled by some authors (see [J1,J2,DR,KK,F1]). However, in [KK], Kirkman and Kuzmanovich found a counterexample to Jategaonkar's conjecture. A counterexample to Tarsy's conjecture was also found in [F1], by providing a tiled D-order in $\mathrm{M}_n(K)$ of global dimension n for every $n \geqslant 6$. It had been expected to find tiled D-orders with finite global dimension larger than n. In [R], Rump found a tiled D-order in $\mathrm{M}_8(K)$ of global dimension 9 from an idea of σ-posets. On the other hand, Jansen and Odenthal found the example mentioned above.

Let R be a basic semiperfect Noetherian ring. Following Ágoston et al. [ADW], we call a primitive idempotent e_n in R *neat* if $\mathrm{Ext}^i_R(S_n, S_n) = 0$ for all $i \geqslant 1$, where S_n is a simple module with $S_n e_n \neq 0$. In Section 1, we shall study some relationships between R and $(1-e_n)R(1-e_n)$. As an application of neat idempotents, in Section 2, we improve the example of Jansen and Odenthal. In Section 3, we shall give another two tiled D-orders having relatively large global dimension. In Section 4, we shall pose two questions on tiled D-orders of finite global dimension, one of which can be considered as an improved version of Jategaonkar's conjecture.

1. Neat idempotents

Throughout this section, let R be a basic semiperfect Noetherian ring with Jacobson radical J, and let $e_1, \ldots, e_n$ be orthogonal primitive idempotents of R with $1 = e_1 + \cdots + e_n$. Put $S_n = e_nR/e_nJ$, $e = 1 - e_n$, and $I = ReR$.

Following Ágoston et al. [ADW], we call a primitive idempotent e_n *neat* if $\mathrm{Ext}^i_R(S_n, S_n) = 0$ for all $i \geqslant 1$. The following proposition is a slight modification of [ADW, Proposition 1]. We give its proof for the reader's convenience.

Proposition 1. *The following statements are equivalent for a primitive idempotent e_n.*

(1) *e_n is neat.*
(2) *Let $\cdots \to P_i \to \cdots \to P_1 \to e_nJ \to 0$ be a minimal projective resolution of e_nJ. Then for each $i \geqslant 1$, $P_i \in \mathrm{add}(eR)$.*
(3) *$e_nJe \otimes_{eRe} eR \cong e_nJ$ by the evaluation map and $\mathrm{Tor}_i^{eRe}(e_nJe, eR) = 0$ for all $i \geqslant 1$.*

(4) *$Re \otimes_{eRe} eR \cong I$, $e_n J e_n = e_n I e_n$, and $\mathrm{Tor}_i^{eRe}(Re, eR) = 0$ for all $i \geqslant 1$.*

Proof. (1) $\Leftrightarrow$ (2). This follows from the fact that for a finitely generated right R-module X, $\mathrm{Ext}_R^i(X, S_n) \neq 0$ if and only if the ith term $P_i(X)$ of a minimal projective resolution of X has a direct summand isomorphic to $e_n R$.

(2) $\Rightarrow$ (3). Applying $- \otimes_R Re$ and then $- \otimes_{eRe} eR$ to the minimal projective resolution of $e_n J$, we obtain the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & P_i e \otimes_{eRe} eR \arrow[r] \arrow[d] & \cdots \arrow[r] & P_1 e \otimes_{eRe} eR \arrow[r] \arrow[d] & e_n J e \otimes_{eRe} eR \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & P_i \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r] & e_n J \arrow[r] & 0.
\end{tikzcd}$$

Since $P_i \in \mathrm{add}(eR)$, we have $P_i e \otimes_{eRe} eR \simeq P_i$. Hence $e_n J e \otimes_{eRe} eR \cong e_n J$ and $\mathrm{Tor}_i^{eRe}(e_n J e, eR) = 0$ for all $i \geqslant 1$.

(3) $\Rightarrow$ (2). Take a minimal projective resolution of $e_n J e$ and apply $- \otimes_{eRe} eR$. Then we have a desired projective resolution of $e_n J$.

(3) $\Leftrightarrow$ (4). Since R is basic, $e_n J e = e_n R e$, so that $Re = eRe \oplus e_n J e$. Let $\varepsilon : e_n J e \otimes_{eRe} eR \to e_n J$ be the evaluation map. Then we can show that ε is monic if and only if $Re \otimes_{eRe} eR \cong I$ and that ε is epic if and only if $e_n J e_n = e_n I e_n$. □

By (4) of Proposition 1, the notion of a neat idempotent is left-right symmetric. As an immediate consequence, we have the following corollary.

Corollary 2. *If e_n is a neat idempotent then $\mathrm{pd}_R(e_n J) = \mathrm{pd}_{eRe}(e_n Re)$ and $\mathrm{pd}_R(J e_n) = \mathrm{pd}_{eRe}(eRe_n)$.*

Next, we consider a converse of Corollary 2, using a projective complex considered in [F2]. We need the following lemma.

Lemma 3. *$I = ReR$ is a maximal ideal if and only if $\mathrm{Ext}_R^1(S_n, S_n) = 0$.*

Proof. In the proof of Proposition 1, it is shown that $\mathrm{Ext}_R^1(S_n, S_n) = 0$ if and only if $e_n J e_n = e_n I e_n$. Since $R/I \cong e_n R e_n / e_n I e_n$, I is maximal if and only if $e_n I e_n = e_n J e_n$. This completes the proof. □

Proposition 4. *Suppose that $\mathrm{Ext}_R^1(S_n, S_n) = 0$ and $\mathrm{pd}_R(e_n J) = s < \infty$. Then e_n is neat if and only if $\mathrm{pd}_{eRe}(e_n Re) \leqslant s$.*

Proof. 'Only if' part follows from Corollary 2. If $\mathrm{pd}_R(e_n J) = 0$ then e_n is neat and $\mathrm{pd}_{eRe}(e_n Re) = 0$. So, we suppose that $s \geqslant 1$ and $u = \mathrm{pd}_{eRe}(e_n Re) \leqslant s$.

Put $L_0 = e_n J$ and let $0 \to K_1 \to P_0 \to L_0 \to 0$ be an exact sequence such that P_0 is a projective cover of L_0. For $i \geqslant 1$, inductively, put $L_i = K_i I$ and let $0 \to K_{i+1} \to P_i \to L_i \to 0$ be an exact sequence such that P_i is a projective cover of L_i. Since I is idempotent, $L_i e_n = L_i I e_n \subset L_i J$. Hence, $P_i \in \operatorname{add}(eR)$ for all $i \geqslant 0$.

First, we show that if $L_i \neq K_i$ for some $i \geqslant 1$ then $\operatorname{pd}_R(L_i) = s$. Let i be the smallest one among such i's. Then

$$0 \to K_i \to P_{i-1} \to \cdots \to P_1 \to P_0 \to L_0 \to 0$$

is exact. If $i > s$ then K_i is isomorphic to a direct summand of P_{i-1}, which contradicts to $L_i \neq K_i$. Hence $i \leqslant s$ and $\operatorname{pd}_R(K_i) = s - i < s$. Since R/I is simple Artinian and $I = \operatorname{ann}(S_n)$ by Lemma 3, K_i/L_i is isomorphic to a direct sum of finite copies of S_n. Hence $\operatorname{pd}_R(K_i/L_i) = s + 1$. Hence by the short exact sequence

$$0 \to L_i \to K_i \to K_i/L_i \to 0$$

we have $\operatorname{pd}_R(L_i) = s$. Repeating the same argument for L_i, we have $\operatorname{pd}_R(L_i) = s$ whenever $L_i \neq K_i$.

Since $L_i e = K_i e$ for all $i \geqslant 1$,

$$\cdots \to P_i e \to \cdots \to P_1 e \to P_0 e \to L_0 e \to 0$$

is a projective resolution of $L_0 e$. Let $i \geqslant u = \operatorname{pd}_{eRe}(e_n Re)$. Then $K_i e$ is eRe-projective, so that $K_i e \otimes_{eRe} eR \cong L_i$ is R-projective. Therefore $K_i = L_i$ because $s \geqslant 1$.

Suppose $L_i \neq K_i$ for some $1 \leqslant i < u$. Take the largest i. Then we have a contradiction that $s = \operatorname{pd}_R(L_i) = u - i < s$. Therefore $K_i = L_i$ for all $i \geqslant 1$, so that

$$\cdots \to P_i \to \cdots \to P_1 \to P_0 \to e_n J \to 0$$

is a minimal projective resolution of $e_n J$ with $P_i \in \operatorname{add}(eR)$ for all $i \geqslant 0$. This completes the proof. □

Remark. In some examples, we can easily compute projective dimensions of $e_n J$ and $e_n Re$ even if their minimal projective resolutions are too complicated. So, Proposition 4 gives a useful criterion for neat idempotents in such examples.

The homology group K_i/L_i of the complex $\{P_i\}$ in the proof of Proposition 4 can be characterized as follows.

Lemma 5. *Let X be a finitely generated right R-module. Put $L_0 = XI$ and let $0 \to K_1 \to P_0 \to L_0 \to 0$ be a short exact sequence with P_0 a projective cover of L_0. For $i \geqslant 1$, inductively, put $L_i = K_i I$ and let $0 \to K_{i+1} \to P_i \to L_i \to 0$ be a short exact sequence with P_i a projective cover of L_i. Then $K_{i+1}/L_{i+1} \cong \operatorname{Tor}_i^{eRe}(Xe, eR)$ for $i \geqslant 1$ and $K_1/L_1 \cong \operatorname{Ker}(Xe \otimes_{eRe} eR \to X)$.*

Proof. Applying $- \otimes_{eRe} eR$ to short exact sequences

$$0 \to K_{i+1}e \to P_i e \to L_i e \to 0$$

we obtain the following commutative diagram with exact columns and rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}_{i+1}(Xe, eR) \arrow[r] & K_{i+1}e \otimes eR \arrow[r] \arrow[d] & P_i e \otimes eR \arrow[r] \arrow[d] & L_i e \otimes eR \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & K_{i+1} \arrow[r] \arrow[d] & P_i \arrow[r] & L_i \arrow[r] \arrow[d] & 0. \\
 & & K_{i+1}/L_{i+1} \arrow[d] & & 0 & \\
 & & 0 & & &
\end{tikzcd}$$

Since $P_i e \otimes eR \cong P_i$ and $K_i e = L_i e$, we obtain the desired isomorphisms by Snake Lemma. □

The following proposition is a refinement of [KK, Proposition 2.6].

Proposition 6. *Suppose that $e_n Re$ (eRe_n) is isomorphic to a right (left) ideal of eRe. Suppose that $\mathrm{Ext}^1_R(S_n, S_n) = 0$, $\mathrm{gl.dim}\, eRe = r + 1 < \infty$, and $\mathrm{pd}_R(e_n J) = s < \infty$. Put $t = \mathrm{pd}_{eRe}(eJe_n)$. Then the following statements hold.*

(1) *If $s + t > r$ then* $\mathrm{gl.dim}\, R = s + t + 2$.
(2) *If $s + t < r$ then* $\mathrm{gl.dim}\, R = r + 1 = \mathrm{gl.dim}\, eRe$.
(3) *If $s + t = r$ then* $\mathrm{gl.dim}\, R \leqslant r + 2$.

Therefore if e_n is neat then $\mathrm{gl.dim}\, R \leqslant 2r + 2$.

Proof. Since $\mathrm{Ext}^1_R(S_n, S_n) = 0$, I is an idempotent maximal ideal and $\mathrm{pd}_R(I) = \mathrm{pd}_R(e_n J) = s$ by Lemma 3.

First, we consider the case $t = 0$. Then e_n is neat by Proposition 1(4), and hence $s = \mathrm{pd}_{eRe}(e_n Re) \leqslant r$. Since eR is eRe-projective, $eJe \otimes_{eRe} eR \cong eJI$ and $\mathrm{pd}_R(eJI) = r$. Consider an exact sequence

$$0 \to eJI \to eJ \to eJ/eJI \to 0. \tag{i}$$

Since $(eJ/eJI)I = 0$, $\mathrm{pd}_R(eJ/eJI) = 0$ or $s + 1$. Thus, if $s = r$ then by (i), we have $\mathrm{pd}_R(eJ) \leqslant r + 1$ and hence $\mathrm{gl.dim}\, R \leqslant r + 2$. If $s < r$ then by (i), we have $\mathrm{pd}_R(eJ) = r$ and hence $\mathrm{gl.dim}\, R = r + 1$.

Now, assume that $t > 0$. Let

$$0 \to Q_r \xrightarrow{f_r} \cdots \to Q_1 \xrightarrow{f_1} Q_0 \xrightarrow{f_0} eJe \to 0$$

be a minimal projective resolution of eJe as a right eRe-module. Put $X_i = \operatorname{Im} f_i$ for $i = 0, 1, \ldots, r$. Applying $- \otimes_{eRe} eR$, we obtain exact sequences

$$0 \to \operatorname{Tor}_i^{eRe}(eJe, eR) \to X_i \otimes_{eRe} eR \to Q_{i-1} \otimes_{eRe} eR \to X_{i-1} \otimes_{eRe} eR \to 0 \tag{ii}$$

for $i = 1, \ldots, r$. Since $\operatorname{pd}_{eRe}(eRe_n) = t \leqslant r$, we have $\operatorname{Tor}_i^{eRe}(eJe, eR) = 0$ for $i \geqslant t$ and $\operatorname{Tor}_t^{eRe}(eRe/eJe, eR) \neq 0$. Hence by (ii), we have $\operatorname{pd}_R(X_{t-1} \otimes_{eRe} eR) = r - t + 1$. Note that $\operatorname{pd}_R(\operatorname{Tor}_t^{eRe}(eRe/eJe, eR)) = s + 1$ because $\operatorname{Tor}_t^{eRe}(eRe/eJe, eR)I = 0$.

(1) Suppose that $s + t > r$. Then $r - t + 1 < s + 1$. Hence by (ii), we have $\operatorname{pd}_R(X_{t-2} \otimes_{eRe} eR) = s + 3, \ldots, \operatorname{pd}_R(X_0 \otimes_{eRe} eR) = s + t + 1$. By an exact sequence

$$0 \to \operatorname{Tor}_1^{eRe}(eRe/eJe, eR) \to eJe \otimes_{eRe} eR \to eJI \to 0, \tag{iii}$$

we have $\operatorname{pd}_R(eJI) = s + t + 1$. Hence by (i), we have $\operatorname{pd}_R(eJ) = s + t + 1$ and hence $\operatorname{gl.dim} R = \sup\{\operatorname{pd}_R(eJ), \operatorname{pd}_R(e_nJ)\} + 1 = s + t + 2$.

(2) Suppose that $s + t < r$. Then $r - t + 1 > s + 1$. Using exact sequences (ii), (iii), and (i), we can show that $\operatorname{gl.dim} R = r + 1$ in a similar way.

(3) This can be shown in a similar way. $\square$

Remark. It follows from [KK, Proposition 2.2] that $\operatorname{gl.dim} eRe \leqslant \operatorname{gl.dim} R + \operatorname{pd}_{eRe}(e_nRe)$. Hence if e_n is neat in R then $\operatorname{gl.dim} eRe \leqslant 2 \cdot \operatorname{gl.dim} R - 1$. (See [ADW, Proposition 2] too.)

As in the proof of Proposition 6(1), we have the following corollary.

Corollary 7. *Suppose that* $\operatorname{Ext}_R^1(S_n, S_n) = 0$ *and* $\operatorname{pd}_R(e_nJ) = s < \infty$. *Let* X *be a finitely generated right* R*-module with* $\operatorname{pd}_{eRe}(Xe) = m < \infty$. *Suppose that there exists* ℓ $(1 \leqslant \ell \leqslant m)$ *such that* $\operatorname{Tor}_i^{eRe}(Xe, eR) = 0$ *if* $i \geqslant \ell$ *and* $\operatorname{Tor}_{\ell-1}^{eRe}(Xe, eR) \neq 0$ *if* $\ell \geqslant 2$ *and that* $m < s + \ell$. *Then* $\operatorname{pd}_R X = s + \ell + 1$.

Lemma 8. *Suppose that* e_n *is neat in* R. *Then for any right* R*-module* X, $\operatorname{Tor}_i^{eRe}(Xe, eR) \cong \operatorname{Tor}_i^R(X, Je_n)$ *for all* $i \geqslant 1$.

Proof. Let

$$\cdots \to P_i \to \cdots \to P_1 \to Je_n \to 0$$

be a minimal projective resolution of Je_n. Since e_n is neat,

$$\cdots \to eP_i \to \cdots \to eP_1 \to eJe_n \to 0$$

is a minimal projective resolution of $eRe_n = eJe_n$. Hence by isomorphisms

$$Xe \otimes_{eRe} eRe_n \cong X \otimes_R Re \otimes_{eRe} eRe_n \cong X \otimes_R Je_n,$$
$$Xe \otimes_{eRe} eP_i \cong X \otimes_R Re \otimes_{eRe} eP_i \cong X \otimes_R P_i,$$

we obtain the desired isomorphisms. □

2. A tiled order having large global dimension

Let D be a discrete valuation ring with a unique maximal ideal πD and quotient field K. Let n ($\geqslant 2$) be an integer, and let λ_{ij} ($1 \leqslant i, j \leqslant n$) be non-negative integers satisfying

$$\lambda_{ik} + \lambda_{kj} \geqslant \lambda_{ij}, \quad \lambda_{ii} = 0 \quad \text{for all } i, j, k \ (1 \leqslant i, j, k \leqslant n)$$

and

$$\lambda_{ij} + \lambda_{ji} > 0 \quad \text{for all } i, j \ (1 \leqslant i, j \leqslant n,\ i \neq j).$$

Then $\Lambda = (\pi^{\lambda_{ij}} D)$ is a D-order in the full matrix ring $\mathrm{M}_n(K)$. Such a D-order Λ is called *tiled*. In what follows, we abbreviate $\Lambda = (\pi^{\lambda_{ij}} D)$ as $\Lambda = (\lambda_{ij})$.

It is shown by Jategaonkar [J2] that for each n, there are only finitely many tiled D-orders in $\mathrm{M}_n(K)$ having finite global dimension. However, it is not known what tiled D-orders have the largest global dimension. In [JO], Jansen and Odenthal found a tiled D-order in $\mathrm{M}_N(K)$ having relatively large global dimension. Namely, for each even $N \geqslant 8$, they constructed a tiled D-order JO_N in $\mathrm{M}_N(K)$ with $\operatorname{gl.dim} JO_N = 2N - 8$.

In this section, we give an inductive construction of JO_N. Namely, we define Γ_N inductively starting from the case $N = 6$. Then we show that $\operatorname{gl.dim} \Gamma_6 = \operatorname{gl.dim} \Gamma_7 = 5$ and $\operatorname{gl.dim} \Gamma_N = 2N - 8$ for *all* $N \geqslant 8$, and we note that $\Gamma_N \cong JO_N$ for even $N \geqslant 8$. A prototype of the inductive construction is summarized in [S].

We begin by recalling some basic facts concerning tiled D-orders. Let $\Lambda = (\lambda_{ij})$ be a tiled D-order in $\mathrm{M}_n(K)$. Then Λ is a basic, semiperfect Noetherian ring of Krull dimension one. The matrix units $e_1 = e_{11}, \dots, e_n = e_{nn}$ are primitive orthogonal idempotents of Λ with $1 = e_1 + \cdots + e_n$. Let J be the Jacobson radical of Λ, which is given by replacing all diagonal entries D of Λ by πD.

The valued quiver $Q(\Lambda) = (Q(\Lambda)_0, Q(\Lambda)_1, v)$ of Λ is defined as follows (see [WR]). $Q(\Lambda)_0 = \{1, \dots, n\}$ is the set of vertices. $Q(\Lambda)_1$ is the set of arrows defined by

$$\alpha : i \to j \in Q(\Lambda)_1 \quad \text{if } \lambda_{jk} + \lambda_{ki} > \lambda_{ji} \quad \text{for all } k \ (1 \leqslant k \leqslant n,\ k \neq i, j).$$

The map v from $Q(\Lambda)_1$ to non-negative integers is defined by

$$v(\alpha) = \begin{cases} \lambda_{ji} & (i \neq j), \\ 1 & (i = j) \end{cases} \quad \text{for any } \alpha : i \to j \in Q(\Lambda)_1.$$

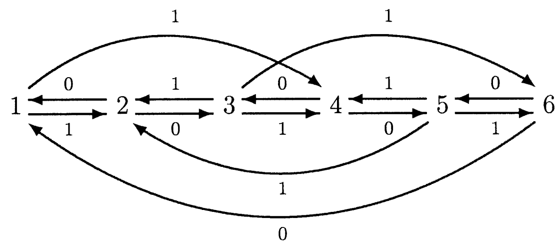
Λ can be recovered by $Q(\Lambda)$. Namely, for each i, j ($1 \leqslant i, j \leqslant n, i \neq j$),

$$\lambda_{ij} = \min\bigl\{v(p) \bigm| p \text{ is a path from } j \text{ to } i \text{ in } Q(\Lambda)\bigr\}$$

where $v(p)$ is the sum of values of all arrows appearing in p. Note that for any path p from j to i in $Q(\Lambda)$ with $v(p) = \lambda_{ij}$, vertices appearing in p are distinct each other.

2.1. *Construction of Γ_N*

Let Γ_6 be the tiled D-order in $\mathrm{M}_6(K)$ having the following valued quiver:



Let $N = 2n$ $(\geqslant 6)$ be an even integer. As an induction hypothesis, we assume that $\Gamma_N = (\gamma_{ij})$ is a tiled D-order in $\mathrm{M}_N(K)$ with the following property:

$$\begin{cases} Q(\Gamma_N) \text{ has arrows } i \to i+1,\ i+1 \to i\ (1 \leqslant i \leqslant N-1), \\ \quad 1 \to N-2,\ 3 \to N,\ N \to 1,\ N-1 \to 2,\ N \to 5, \\ \quad N-4 \to 1,\ 4 \to N-1; \\ \text{for each } \alpha : i \to j \in Q(\Gamma_N)_1, \\ \quad \text{if } i \text{ is even then } j \text{ is odd and } v(\alpha) = 0, \\ \quad \text{if } i \text{ is odd then } j \text{ is even and } v(\alpha) = 1. \end{cases} \tag{$*$}$$

Note that Γ_6 has this property.

2.2. *Step of Γ_{N+1}*

We make a new valued quiver Q' by adding a new vertex $N+1$ and four valued arrows $N \xrightarrow{0} N+1$, $N+1 \xrightarrow{1} N$, $2 \xrightarrow{0} N+1$, and $N+1 \xrightarrow{1} 4$ to the valued quiver $Q(\Gamma_N)$. Then for any i, j $(1 \leqslant i, j \leqslant N)$, put

$$\gamma_{i,N+1} = \min\{v(p) \mid p \text{ is a path from } N+1 \text{ to } i \text{ in } Q'\},$$

$$\gamma_{N+1,j} = \min\{v(p) \mid p \text{ is a path from } j \text{ to } N+1 \text{ in } Q'\},$$

and put $\Gamma_{N+1} = (\gamma_{ij})_{1 \leqslant i,j \leqslant N+1}$, where $\gamma_{N+1,N+1} = 0$.

Claim 1. *Γ_{N+1} is a tiled D-order in* $\mathrm{M}_{N+1}(K)$.

Proof. By the definition of $\gamma_{i,N+1}$ and $\gamma_{N+1,j}$, it is easily verified that $\gamma_{ij} + \gamma_{j,N+1} \geqslant \gamma_{i,N+1}$ and $\gamma_{N+1,i} + \gamma_{ij} \geqslant \gamma_{N+1,j}$ for all i, j $(1 \leqslant i, j \leqslant N)$. So, it is sufficient to verify that $\gamma_{i,N+1} + \gamma_{N+1,j} \geqslant \gamma_{ij}$, $\gamma_{i,N+1} + \gamma_{N+1,i} > 0$ for all $1 \leqslant i, j \leqslant N$. Note that $\gamma_{i,N+1} + \gamma_{N+1,j}$, is the smallest value of paths p from j

to i via $N+1$ in Q'. Such paths must have one of subpaths $N \xrightarrow{0} N+1 \xrightarrow{1} N$, $N \xrightarrow{0} N+1 \xrightarrow{1} 4$, $2 \xrightarrow{0} N+1 \xrightarrow{1} N$, or $2 \xrightarrow{0} N+1 \xrightarrow{1} 4$. In each case, there is a path p' from j to i in $Q(\Gamma_N)$ with $v(p') = v(p)$. For example, $2 \xrightarrow{0} N+1 \xrightarrow{1} N$ can be replaced by $2 \xrightarrow{0} 3 \xrightarrow{1} N$. Hence $\gamma_{i,N+1} + \gamma_{N+1,j} = v(p) = v(p') \geqslant \gamma_{ij}$. Moreover, since the values of the above four subpaths are 1, we have $\gamma_{i,N+1} + \gamma_{N+1,i} > 0$. $\square$

Claim 2. $Q(\Gamma_{N+1}) = Q'$.

Proof. Since $\gamma_{N+1,N} + \gamma_{N,N+1} = 1$, $N+1 \to N+1 \notin Q(\Gamma_{N+1})_1$. Therefore, since $Q(\Gamma_N)$ has no loop, so does $Q(\Gamma_{N+1})$.

Let $\alpha : i \to j \in Q(\Gamma_{N+1})_1$. Then $\gamma_{jk} + \gamma_{ki} > \gamma_{ji}$ for all k $(1 \leqslant k \leqslant N+1$, $k \neq i, j)$. If $1 \leqslant i, j \leqslant N$, then clearly we have $\alpha \in Q(\Gamma_N)_1 \subset Q'_1$. Assume that $i = N+1$ and $j \neq 4, N$. Since $N+1 \to 4$ or $N+1 \to N$ is the first arrow of any path from $N+1$ to j in Q', we have $\gamma_{j,N+1} = \gamma_{j4} + \gamma_{4,N+1}$ or $\gamma_{jN} + \gamma_{N,N+1}$, a contradiction. Hence $j = 4$ or N. Similarly $j = N+1$ implies $i = 2$ or N. Thus $\alpha \in Q'_1$.

Conversely, let $\alpha : i \to j \in Q'_1$. First consider the case of $\alpha \in Q(\Gamma_N)_1$. As in the proof of Claim 1, there is a path p' in $Q(\Gamma_N)$ with $\gamma_{j,N+1} + \gamma_{N+1,i} = v(p')$. Since $\alpha \in Q(\Gamma_N)_1$, $v(p') > \gamma_{ji}$ so that $\gamma_{jk} + \gamma_{ki} > \gamma_{ji}$ for all $k \neq i, j$. Hence $\alpha \in Q(\Gamma_{N+1})_1$. In the case of $\alpha = N \to N+1$, assume that $\gamma_{N+1,k} + \gamma_{kN} = \gamma_{N+1,N}$ for some $k \neq N, N+1$. Since $\gamma_{N+1,k} = \gamma_{N+1,2} + \gamma_{2k}$ or $\gamma_{N+1,N} + \gamma_{Nk}$, we have by the property $(*)$: $0 = \gamma_{N+1,N} = \gamma_{N+1,2} + \gamma_{2k} + \gamma_{kN}$ or $\gamma_{N+1,N} + \gamma_{Nk} + \gamma_{kN} > 0$, a contradiction. Hence $\alpha \in Q(\Gamma_{N+1})_1$. Similarly we can show the remaining cases. $\square$

2.3. *Step of Γ_{N+2}*

We make a new valued quiver Q'' by adding a new vertex 0 and five valued arrows $0 \xrightarrow{0} 1$, $1 \xrightarrow{1} 0$, $0 \xrightarrow{0} N-1$, $N-3 \xrightarrow{1} 0$, and $N+1 \xrightarrow{1} 0$ to the valued quiver $Q(\Gamma_{N+1})$. Then for any i, j $(1 \leqslant i, j \leqslant N+1)$, put

$$\gamma_{i0} = \min\{v(p) \mid p \text{ is a path from 0 to } i \text{ in } Q''\},$$

$$\gamma_{0j} = \min\{v(p) \mid p \text{ is a path from} j \text{ to 0 in } Q''\},$$

and put $\Gamma_{N+2} = (\gamma_{ij})_{0 \leqslant i, j \leqslant N+1}$ where $\gamma_{0,0} = 0$.

As in the step of Γ_{N+1}, we can show the following fact.

Claim 3. *Γ_{N+2} is a tiled D-order in* $\mathrm{M}_{N+2}(K)$ *with* $Q(\Gamma_{N+2}) = Q''$.

We shift the names of vertices from $0, 1, \ldots, N+1$ to $1, 2, \ldots, N+2$, respectively. Let u be a diagonal matrix in $\mathrm{M}_{N+2}(K)$ with the (i,i)-entry π if i is odd and 1 otherwise. Then $u\Gamma_{N+2}u^{-1}$ is a tiled D-order with the property $(*)$. Thus, we have constructed Γ_N by induction.

For even $N \geqslant 8$, one can verify that $\Gamma_N \cong JO_N$ by inner automorphism given by a permutation and change of values.

Note that there is a symmetry of Γ_N for even N. Namely, applying the permutation

$$\begin{pmatrix} 1 & 2 & \cdots & N \\ N & N-1 & \cdots & 1 \end{pmatrix}$$

to the vertices of Γ_N, we obtain the opposite ring of Γ_N.

Next, we compute global dimension of Γ_N by induction. In the rest of this section, we use the following notations for Γ_N $(N \geqslant 6)$. $J(N)$ (or J) denotes the Jacobson radical of Γ_N. P_i (respectively P_i') denotes the indecomposable projective right (respectively left) Γ_N-module $e_i\Gamma_N$ (respectively $\Gamma_N e_i$) for $1 \leqslant i \leqslant N$. S_i (respectively S_i') denotes the simple right (respectively left) Γ_N-module P_i/e_iJ (respectively P_i'/Je_i) for $1 \leqslant i \leqslant N$. We decompose Γ_{N+1} and Γ_{N+2} as follows:

$$\Gamma_{N+1} = \left(\begin{array}{c|c} \Gamma_N & A \\ \hline B & 0 \end{array}\right) \qquad \Gamma_{N+2} = \left(\begin{array}{c|c} 0 & X \\ \hline Y & \Gamma_{N+1} \end{array}\right).$$

In order to obtain elementary short exact sequences, which will be used in the induction steps, we need the following lemma.

Lemma 9. *Let N $(\geqslant 6)$ be even. Then for $\Gamma_N = (\gamma_{ij})$, the following statements hold.*

(1) $\gamma_{2j} \leqslant \gamma_{Nj}$ *for all* j $(1 \leqslant j \leqslant N-1)$.
(2) $\gamma_{i4} \leqslant \gamma_{iN}$ *for all* i $(2 \leqslant i \leqslant N-1)$.

Proof. (1) Let p be a γ_{Nj}-path from j to N in $Q(\Gamma_N)$. Then the last arrow of p is $3 \to N$ or $N-1 \to N$. Since there are arrows $3 \to 2$ and $N-1 \to 2$, $\gamma_{Nj} = \gamma_{23} + \gamma_{3j}$ or $\gamma_{2,N-1} + \gamma_{N-1,j} \geqslant \gamma_{2j}$.

(2) Let q be a γ_{iN}-path from N to i in $Q(\Gamma_N)$. Then the first arrow α of q is one of $N \to 1$, $N \to 5$, or $N \to N-1$. Since there are arrows $4 \to 5$ and $4 \to N-1$, we can show that $\gamma_{i4} \leqslant \gamma_{iN}$ as in the proof of (1), in the case of $\alpha = N \to 5$ or $N \to N-1$. Suppose that $\alpha = N \to 1$ and $\gamma_{iN} < \gamma_{i4}$. Then, since there is a path $4 \xrightarrow{0} 3 \xrightarrow{1} N$, we obtain that $\gamma_{i4} = \gamma_{iN} + 1$. Hence we have a γ_{i4}-path

$$4 \to 3 \to N \to 1 \to \cdots \to i.$$

Let a be the terminal vertex of the arrow starting at 1 above. Then a is 2 or $N-2$. However, replacing the subpath $3 \to N \to 1$ by $3 \to 2 \to 1$, we obtain another γ_{i4}-path from 4 to i. Hence we conclude that $a = N-2$. Hence $4 \xrightarrow{0} 3 \xrightarrow{1} N \xrightarrow{0} 1 \xrightarrow{1} N-2$ is a $\gamma_{N-2,4}$-path whose value is 2. However, there is a path $4 \xrightarrow{0} N-1 \xrightarrow{1} N-2$, a contradiction. This completes the proof. $\square$

Lemma 10. *Let N ($\geqslant 6$) be even. Then there exist the following short exact sequences of Γ_N-modules.*

(1) $0 \to P_2 \to B \to S_N \to 0$.
(2) $0 \to e_N J(N) \to P_2 \oplus P_N \to B \to 0$.
(3) $0 \to \Gamma_N e_4 \to A \to S \to 0$ *where* $S \cong S'_N$ *if* $N = 8$ *and* $S \cong S'_1 \oplus S'_N$ *if* $N \neq 8$.
(4) $0 \to J(N)e_N \to P'_4 \oplus P'_N \to A_1 \to 0$ *where* $A_1 = A$ *if* $N = 8$ *and* $A/A_1 \cong S'_N$ *if* $N \geqslant 10$.

Proof. (1), (2). Since $N+1^- = \{2, N\}$, $\gamma_{N+1,j} = \gamma_{N+1,2} + \gamma_{2j} = \gamma_{2j}$ or $\gamma_{N+1,j} = \gamma_{N+1,N} + \gamma_{Nj} = \gamma_{Nj}$ for $1 \leqslant j \leqslant N-1$. Hence by Lemma 9(1), $\gamma_{N+1,j} = \gamma_{2j}$ for $1 \leqslant j \leqslant N-1$. Since there is a path $N \xrightarrow{0} 1 \xrightarrow{1} 2$, we have $\gamma_{2N} = 1$. Moreover $\gamma_{N+1,N} = 0$. Hence we obtain exact sequences of (1) and (2) with usual maps.

(3), (4). Since $N+1^+ = \{4, N\}$, $\gamma_{i,N+1} = \gamma_{i4} + \gamma_{4,N+1} = \gamma_{i4} + 1$ or $\gamma_{i,N+1} = \gamma_{iN} + \gamma_{N,N+1} = \gamma_{iN} + 1$ for $2 \leqslant i \leqslant N-1$. Hence by Lemma 9(2), $\gamma_{i,N+1} = \gamma_{i4} + 1$ for $2 \leqslant i \leqslant N-1$. Since there is a path $4 \xrightarrow{0} N-1 \xrightarrow{1} N$, we have $\gamma_{4N} = 1$. If $N = 8$, there is an arrow $4 \xrightarrow{0} 1$, so $\gamma_{14} = 0$. If $N \geqslant 10$, there is no arrow from 4 to 1, so $\gamma_{14} = 1$. Hence we obtain exact sequences of (3) and (4) with usual maps. $\square$

2.4. Computation of gl.dim Γ_N

First, we compute gl.dim Γ_6. Note that

$$\Gamma_6 = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Then there exist the following exact sequences:

$$0 \to e_6 J \to P_2 \oplus P_6 \to e_1 J \to 0,$$

$$0 \to P_4 \to P_3 \oplus P_5 \to e_6 J \to 0,$$

$$0 \to e_6 J \to P_4 \oplus P_6 \to e_5 J \to 0,$$

$$0 \to e_2 J\ (\cong e_4 J) \to P_2 \oplus P_4 \to e_3 J \to 0,$$

$$0 \to e_1 J \to P_1 \oplus e_6 J \to e_2 J \to 0.$$

Hence we obtain that $\operatorname{pd} e_1 J = 2$, $\operatorname{pd} e_2 J = 3$, $\operatorname{pd} e_3 J = 4$, $\operatorname{pd} e_4 J = 3$, $\operatorname{pd} e_5 J = 2$, $\operatorname{pd} e_6 J = 1$, so that $\operatorname{gl.dim} \Gamma_6 = \operatorname{pd} J + 1 = 5$. Note that

$$\Gamma_6 = \left(\begin{array}{cccccc|c} 0&0&1&1&1&0&1\\ 1&0&1&1&1&1&2\\ 1&0&0&0&1&1&1\\ 1&1&1&0&1&1&1\\ 1&1&1&0&0&0&1\\ 2&1&1&1&1&0&1\\ \hline 1&0&1&1&1&0&0 \end{array}\right) = \left(\begin{array}{c|c} \Gamma_6 & A\\ \hline B & 0 \end{array}\right).$$

Then we obtain the following minimal projective resolutions of $e_7 J$ and A:

$$0 \to P_4 \to P_3 \oplus P_5 \to P_2 \oplus P_6 \to e_7 J \to 0,$$
$$0 \to P_5' \to P_4' \oplus P_6' \to A \to 0.$$

Hence e_7 is neat in Γ_7, $\operatorname{pd} e_7 J = 2$, and $\operatorname{pd} A = 1$. It follows from Proposition 6(2) that $\operatorname{gl.dim} \Gamma_7 = \operatorname{gl.dim} \Gamma_6 = 5$.

Note that

$$\Gamma_8 = \left(\begin{array}{c|ccccccc} 0&1&1&1&1&2&1&1\\ \hline 0&0&0&1&1&2&0&1\\ 1&1&0&1&1&1&1&2\\ 1&1&0&0&0&1&1&1\\ 1&1&1&1&0&1&1&1\\ 0&1&1&1&0&0&0&1\\ 1&2&1&1&1&1&0&1\\ 1&1&0&1&1&1&0&0 \end{array}\right) = \left(\begin{array}{c|c} 0 & X\\ \hline Y & \Gamma_7 \end{array}\right).$$

Then we have a minimal projective resolution

$$0 \to P_4' \to P_3' \oplus P_5' \to P_2' \oplus P_6' \to J e_1 \to 0.$$

Hence e_1 is neat in Γ_8 and $\operatorname{pd} J e_1 = 2$. One can compute $\operatorname{pd} X = 4$ directly or using exact sequences $0 \to K_1/L_1 \to Xe \otimes_{\Gamma_6} e\Gamma_7 \to XI \to 0$ (by Lemma 5) and $0 \to XI \to X \to X/XI \to 0$. It follows from Proposition 6(1) that $\operatorname{gl.dim} \Gamma_8 = 8$.

Note that since $\operatorname{pd}_{\Gamma_6} A = 1$, $\operatorname{Tor}_i^{\Gamma_6}(Xe, A) = 0$ for all $i \geqslant 1$.

Let $N = 2n$ ($\geqslant 8$) and as an induction hypothesis, assume the following:

$$\operatorname{gl.dim} \Gamma_N = 2N - 8,$$
$$\exists\, 0 \to P_{n+1} \to P_n \oplus P_{n+2} \to \cdots \to P_3 \oplus P_{N-1} \to e_N J(N) \to 0 \quad \text{(exact)},$$
$$\operatorname{pd} e_1 J(N) = \operatorname{pd} J(N) e_N = 3n - 8,$$
$$\operatorname{Tor}_{2n-7}^{\Gamma_N}(e_1 J(N), A) \neq 0, \qquad \operatorname{Tor}_i^{\Gamma_N}(e_1 J(N), A) = 0 \quad \text{for } i \geqslant 2n - 6.$$

2.5. Step of Γ_8

It is sufficient to show that Γ_8 satisfies the last condition above. Since e_1 is neat in Γ_8 and e_7 is neat in Γ_7, it follows from Lemma 8 that

$$\mathrm{Tor}_i^{\Gamma_8}(e_1 J, Je_8) \cong \mathrm{Tor}_i^{\Gamma_6}(Xe, A) = 0$$

for all $i \geqslant 1$. Hence by Lemma 10(4), $\mathrm{Tor}_i^{\Gamma_8}(e_1 J, A) = 0$ for all $i \geqslant 2$. We can verify that $\mathrm{Tor}_1^{\Gamma_8}(e_1 J, A) \neq 0$ by Lemma 5 and [JO, Lemma 1.10].

2.6. Step of Γ_{N+1}

By Lemma 10(2) and the assumption, we have a minimal projective resolution of B:

$$0 \to P_{n+1} \to P_n \oplus P_{n+2} \to \cdots \to P_3 \oplus P_{N-1} \to P_2 \oplus P_N \to B \to 0.$$

Therefore, by Lemma 10(3), $\mathrm{Tor}_i^{\Gamma_N}(B, A) \cong \mathrm{Tor}_i^{\Gamma_N}(B, S) = 0$ for all $i \geqslant 1$. Applying $- \otimes_{\Gamma_N} e\Gamma_{N+1}$ to the projective resolution of B, we obtain an exact sequence

$$\begin{aligned} 0 \to\ & P_{n+1} \to P_n \oplus P_{n+2} \to \cdots \to P_3 \oplus P_{N-1} \to P_2 \oplus P_N \\ \to\ & B \otimes_{\Gamma_N} e\Gamma_{N+1} \to 0 \end{aligned}$$

where $e = 1 - e_{N+1}$. Then, since $\min\{\gamma_{2,N+1}, \gamma_{N,N+1}\} = 1$, we obtain a minimal projective resolution of $e_{N+1} J(N+1)$:

$$\begin{aligned} 0 \to\ & P_{n+1} \to P_n \oplus P_{n+2} \to \cdots \to P_3 \oplus P_{N-1} \to P_2 \oplus P_N \\ \to\ & e_{N+1} J(N+1) \to 0. \end{aligned}$$

Therefore, e_{N+1} is neat in Γ_{N+1} and $\mathrm{pd}\, e_{N+1} J(N+1) = n - 1$. It follows from Lemma 10(4) that $\mathrm{pd}_{\Gamma_N} A = \mathrm{pd}\, J(N)e_N + 1 = 3n - 7$. Thus it follows from Proposition 6(1) that $\mathrm{gl.dim}\, \Gamma_{N+1} = 2(N+1) - 8$.

2.7. Step of Γ_{N+2}

By the symmetry, as in Step of Γ_{N+1}, we obtain a minimal projective resolution of $B' = {}^t(\gamma_{1,0}, \ldots, \gamma_{N,0})$:

$$0 \to P'_n \to P'_{n+1} \oplus P'_{n-1} \to \cdots \to P'_{N-2} \oplus P'_2 \to P'_{N-1} \oplus P'_1 \to B' \to 0.$$

Therefore by Lemma 10(1), $\mathrm{Tor}_i^{\Gamma_N}(\Gamma_{N+1}e, B') \cong \mathrm{Tor}_i^{\Gamma_N}(S_N, B') = 0$ for all $i \geqslant 1$. Applying $\Gamma_{N+1}e \otimes_{\Gamma_N} -$ to the projective resolution of B', we obtain an exact sequence:

$$\begin{aligned} 0 \to\ & P'_n \to P'_{n+1} \oplus P'_{n-1} \to \cdots \to P'_{N-2} \oplus P'_2 \to P'_{N-1} \oplus P'_1 \\ \to\ & \Gamma_{N+1}e \otimes_{\Gamma_N} B' \to 0. \end{aligned}$$

Then, since $\min\{\gamma_{N+1,1}, \gamma_{N+1,N-1}\} = 1$, the above sequence gives a projective resolution of Y. By the symmetry, as in Lemma 10(3), we obtain an exact sequence

$$0 \to e_{N-3}\Gamma_{N+1} \to X \to T \to 0$$

where T is of finite length with composition factors S_1, S_N, S_{N+1}. Therefore

$$\mathrm{Tor}_i^{\Gamma_{N+1}}(X, Y) \cong \mathrm{Tor}_i^{\Gamma_{N+1}}(T, Y) = 0$$

for all $i \geqslant 1$. Hence we obtain a projective resolution of $J(N+2)e_0$:

$$\begin{aligned} 0 \to\ & P'_n \to P'_{n+1} \oplus P'_{n-1} \to \cdots \to P'_{N-2} \oplus P'_2 \to P'_{N-1} \oplus P'_1 \\ \to\ & J(N+2)e_0 \to 0. \end{aligned}$$

Hence e_0 is neat in Γ_{N+2} and $\mathrm{pd}\, J(N+2)e_0 = n-1$. It follows from Corollary 7 and the induction hypothesis that $\mathrm{pd}\, e_1 J(N+1) = 3n-6$. Hence $\mathrm{pd}\, X = \mathrm{pd}\, T = \mathrm{pd}\, e_1 J(N+1) + 1 = 3n-5$. Thus, by Proposition 6(1), $\mathrm{gl.dim}\, \Gamma_{N+2} = 2(N+2) - 8$.

In order to complete the induction, we need to show that

$$\mathrm{Tor}_{2(n+1)-7}^{\Gamma_{N+2}}\big(e_0 J(N+2), A(N+2)\big) \neq 0$$

and

$$\mathrm{Tor}_i^{\Gamma_{N+2}}\big(e_0 J(N+2), A(N+2)\big) = 0 \quad \text{for all } i \geqslant 2(n+1) - 6.$$

Since e_{N+1} is neat in Γ_{N+1} and e_0 is neat in Γ_{N+2}, by Lemmas 8 and 10(3), we obtain the following isomorphisms:

$$\begin{aligned} \mathrm{Tor}_i^{\Gamma_N}\big(e_1 J(N), A\big) &\cong \mathrm{Tor}_{i+1}^{\Gamma_N}(Xe, A) \\ &\cong \mathrm{Tor}_{i+1}^{\Gamma_{N+1}}\big(X, J(N+1)e_{N+1}\big) \\ &\cong \mathrm{Tor}_{i+1}^{\Gamma_{N+2}}\big(e_0 J(N+2), J(N+2)e_{N+1}\big) \\ &\cong \mathrm{Tor}_{i+2}^{\Gamma_{N+2}}\big(e_0 J(N+2), A(N+2)\big) \end{aligned}$$

for all $i \geqslant n-1$. This completes the induction.

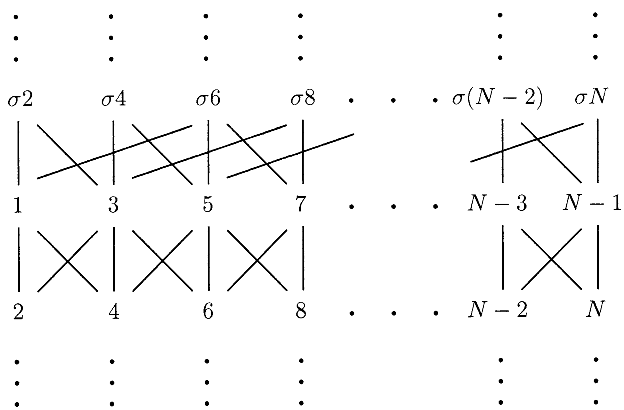
3. Another tiled orders having large global dimension

In this section, we give two tiled orders having relatively large global dimension. In [R], Rump found a tiled D-order R_8 in $\mathrm{M}_8(K)$ of global dimension 9, which is larger than $\mathrm{gl.dim}\, JO_8 = 8$. R_8 is also a modification of [F1, Example 2.5] by means of σ-posets (see [R] for definition). The following example may be a natural extension of [F1, Example 2.5] in this direction.

Example 1. Let $N = 2n$ ($\geqslant 6$) be an even integer. Let $Q = (Q_0, Q_1, v)$ be the valued quiver such that $Q_0 = \{1, 2, \dots, N\}$ is the set of vertices, Q_1 is the set of the following $6n - 5$ arrows:

$$2k-1 \to 2k, \quad 2k \to 2k-1 \quad (1 \leqslant k \leqslant n),$$
$$2k+1 \to 2k, \quad 2k \to 2k+1, \quad 2k+2 \to 2k-1 \quad (1 \leqslant k \leqslant n-1),$$
$$2k-1 \to 2k+4 \quad (1 \leqslant k \leqslant n-2),$$

and that for $\alpha : i \to j \in Q_1$, $v(\alpha) = 1$ (if i is odd) and 0 (if i is even). Let Λ_N be the tiled D-order defined by Q. The σ-poset of Λ_N is as follows:



Then gl.dim $\Lambda_N = 3n - 3$. One can execute its computation as in Section 2.

The following tiled D-order in $\mathrm{M}_8(K)$ has global dimension 10. By experiments, we guess that its inductive extension exceeds Γ_N in global dimension.

Example 2. Let

$$\Lambda = \begin{pmatrix} 0 & 1 & 1 & 2 & 2 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

be a tiled D-order in $\mathrm{M}_8(K)$. Then gl.dim $\Lambda = 10$.

4. Remarks

In this section, we pose two questions on tiled D-orders of finite global dimension.

As pointed out in [ADW, Example 4], there is a path algebra A of finite global dimension with no neat primitive idempotent. However, in the class of tiled D-orders, we do not know such examples.

Question 1. Does any tiled D-order of finite global dimension have a neat primitive idempotent?

Question 1 can be considered as an improved version of Jategaonkar's conjecture. If Question 1 is true, using Proposition 6, we can show that $3 \cdot 2^{n-5}$ is a upper bound of finite global dimensions of tiled D-orders in $\mathrm{M}_n(K)$ for $n \geqslant 6$. Using computer, we have verified the upper bound is 6 when $n = 6$.

For a tiled D-order $\Lambda = (\lambda_{ij})$ in $\mathrm{M}_n(K)$, put $d(\Lambda) = \sum_{1 \leqslant i,j \leqslant n} \lambda_{ij}$. We call $d(\Lambda)$ *depth* of Λ. It is known that Λ is hereditary if and only if $d(\Lambda) = \frac{1}{2}n(n-1)$, which is the smallest depth among tiled D-orders in $\mathrm{M}_n(K)$.

Question 2. If gl.dim $\Lambda < \infty$, is then $d(\Lambda) \leqslant \frac{1}{6}(n+1)n(n-1)$?

Let Ω_n be the tiled D-order in $\mathrm{M}_n(K)$ given by the following valued quiver:

$$1 \underset{1}{\overset{0}{\rightleftarrows}} 2 \underset{1}{\overset{0}{\rightleftarrows}} 3 \underset{1}{\overset{0}{\rightleftarrows}} \cdots \underset{1}{\overset{0}{\rightleftarrows}} n-1 \underset{1}{\overset{0}{\rightleftarrows}} n.$$

Then gl.dim $\Omega_n = 2$ and $d(\Omega_n) = \frac{1}{6}(n+1)n(n-1)$.

Using [J2, Lemma 2.2], one can show that if Question 1 is true then so is Question 2. Moreover, if Question 2 is true, then it follows from [J2, Theorem 2.11] that Ω_n is a unique (up to isomorphism) basic tiled D-order in $\mathrm{M}_n(K)$ of finite global dimension having the largest depth.

References

[ADW] I. Ágoston, V. Dlab, T. Wakamatsu, Neat algebras, Comm. Algebra 19 (2) (1991) 433–442.

[DR] J.A. De La Peña, A. Raggi-Cárdenas, On the global dimension of algebras over regular local rings, Illinois J. Math. 32 (3) (1988) 520–533.

[F1] H. Fujita, Tiled orders of finite global dimension, Trans. Amer. Math. Soc. 322 (1990) 329–341; Erratum: Trans. Amer. Math. Soc. 327 (1991) 919–920.

[F2] H. Fujita, A construction of tilting modules associated with a simple module, J. Algebra 145 (1992) 224–230.

[JO] W.S. Jansen, C.J. Odenthal, A tiled order having large global dimension, J. Algebra 192 (1997) 572–591.

[J1] V.A. Jategaonkar, Global dimension of triangular orders over a discrete valuation ring, Proc. Amer. Math. Soc. 38 (1973) 8–14.

[J2] V.A. Jategaonkar, Global dimension of tiled orders over a discrete valuation ring, Trans. Amer. Math. Soc. 196 (1974) 313–330.

[KK] E. Kirkman, J. Kuzmanovich, Global dimension of a class of tiled orders, J. Algebra 127 (1989) 57–72.

[R] W. Rump, Discrete posets, cell complexes, and the global dimension of tiled orders, Comm. Algebra 24 (1) (1996) 55–107.

[S] H. Suzuki, On a tiled order having large global dimension, Master thesis, University of Tsukuba, 1999 (in Japanese).

[T1] R.B. Tarsy, Global dimension of orders, Trans. Amer. Math. Soc. 151 (1970) 335–340.

[T2] R.B. Tarsy, Global dimensions of triangular orders, Proc. Amer. Math. Soc. 28 (2) (1971) 423–426.

[WR] A. Wiedemann, K.W. Roggenkamp, Path orders of global dimension two, J. Algebra 80 (1983) 113–133.